Cohomological construction of quantized universal enveloping algebras

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Abstract. Given an associative algebra A, and the category, $\mathcal{C}$, of its finite dimensional modules, additional structures on the algebra A induce corresponding ones on the category $\mathcal{C}$. Thus, the structure of a rigid quasi-tensor (braided monoidal) category on Rep_A is induced by an algebra homomorphism $A \to A \otimes A$ (comultiplication), coassociative up to conjugation by $\Phi \in A^{\otimes 3}$ (associativity constraint) and cocommutative up to conjugation by $\mathcal{R} \in A^{\otimes 2}$ (commutativity constraint), together with an antiautomorphism (antipode), S, of A satisfying the certain compatibility conditions. A morphism of quasi-tensor structures is given by an element $F \in A^{\otimes 2}$ with suitable induced actions on Φ, $\mathcal{R}$ and S. Drinfeld defined such a structure on $A = U(\mathcal{G})[[h]]$ for any semisimple Lie algebra $\mathcal{G}$ with the usual comultiplication and antipode but nontrivial $\mathcal{R}$ and Φ and proved that the corresponding quasi-tensor category is isomomorphic to the category of representations of the Drinfeld-Jimbo (DJ) quantum universal enveloping algebra (QUE), $U_h(\mathcal{G})$.

In the paper we give a direct cohomological construction of the F which reduces Φ to the trivial associativity constraint, without any assumption on the prior existence of a strictly coassociative QUE. Thus we get a new approach to the DJ quantization. We prove that F can be chosen to satisfy some additional invariance conditions under (anti)automorphisms of $U(\mathcal{G})[[h]]$, in particular, F gives an isomorphism of rigid quasi-tensor categories. Moreover, we prove that for pure imaginary values of the deformation parameter, the elements F, R and Φ can be chosen to be formal unitary operators on the second and third tensor powers of the regular representation of the Lie group associated to $\mathcal{G}$ with Φ depending only on even powers of the deformation parameter. In addition, we consider some extra properties of these elements and give their interpretation in terms of additional structures on the relevant categories.

§0 Introduction.

The original interest in quantum groups and quantized universal enveloping algebras (QUE) came from the existence of a universal R-matrix satisfying the Yang-Baxter equation. From the point of view of representation theory the Yang-Baxter equation is a consequence of the more fundamental hexagon identities, which are the basic identities for the braiding in a braided monoidal (quasi-tensor) category. However from the categorical perspective there is something unnatural in the standard axioms for quantum groups since there is a second basic operator, the

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associativity constraint, which is restricted to be the identity. A more natural, although technically more involved, set of axioms was introduced by Drinfeld in [Dr1, Dr2] for the algebraic structure which he called "quasi-triangular quasi-Hopf algebra." The axioms for such structures are precisely those required in order that the category of finite dimensional representations be a rigid braided monoidal category, without restricting the associativity constraint to be the identity.

We review some basic definitions. A quasi-bialgebra is an associative algebra, A, with comultiplication $\Delta : A \to A \otimes A$, which is not necessarily coassociative. However there is given an invertible element of the triple tensor product, $\Phi \in A^{\otimes 3}$, which expresses the relation between the two iterated comultiplications by the formula

$$\Phi(\Delta \otimes id)\Delta(a) = (id \otimes \Delta)\Delta(a)\Phi \quad \text{for} \quad a \in A. \tag{1}$$

An additional condition on Φ is the identity in $A^{\otimes 4}$,

$$(id^{\otimes 2} \otimes \Delta)(\Phi) \cdot (\Delta \otimes id^{\otimes 2})(\Phi) = (1 \otimes \Phi) \cdot (id \otimes \Delta \otimes id)(\Phi) \cdot (\Phi \otimes 1). \tag{2}$$

This equation comes from requiring the pentagon identity in the category of finite dimensional representations of A as explained in the next section. In a quasi-triangular quasi-bialgebra we are given a so-called R-matrix, $\mathcal{R}$, which is an invertible element of the double tensor product, $A^{\otimes 2}$, expressing the relation between the comultiplication and the opposite comultiplication, $\Delta^{\mathrm{op}} = \sigma \circ \Delta$ for $\sigma(a \otimes b) = b \otimes a$, by the formula

$$\mathcal{R}\Delta(a) = \Delta^{\mathrm{op}}(a)\mathcal{R}, \quad \text{for} \quad a \in A. \tag{3}$$

The compatibility conditions between $\mathcal{R}$ and Φ are given by the identities in $A^{\otimes 3}$,

$$(\Delta \otimes id)\mathcal{R} = \Phi^{312}\mathcal{R}^{13}(\Phi^{132})^{-1}\mathcal{R}^{23}\Phi \tag{4a}$$

$$(id \otimes \Delta)\mathcal{R} = (\Phi^{231})^{-1}\mathcal{R}^{13}\Phi^{213}\mathcal{R}^{12}\Phi^{-1}. \tag{4b}$$

These two equations correspond to two commutative hexagons diagrams which, according to the MacLane coherence theorem, together with the pentagon diagram

corresponding to (3) generate all the relations involving the associativity constraint and the commutativity constraint in a quasi-tensor category. A quasi-bialgebra with an antipode (see §1) is called a quasi-Hopf algebra.

In the context of deformation theory, Drinfeld proved two important results about the existence and uniqueness of such structures. Let $\mathcal{G}$ be a Lie algebra over a field K of characteristic 0, $K[[h]]$ be the ring of formal power series with coefficients in K, and $U(\mathcal{G})[[h]] = U(\mathcal{G}) \otimes_K K[[h]]$, the completed tensor product. The first result says that for any symmetric $\mathcal{G}$-invariant element $t \in \mathcal{G}^{\otimes 2}$ there exists a $\mathcal{G}$-invariant element $\Phi_h \in U(\mathcal{G})^{\otimes 3}[[h]]$ satisfying the pentagon identity and such that together with $\mathcal{R}_h = e^{\pi \imath h t}$ it satisfies the hexagon identities. These elements define a deformation (quantization) of the universal enveloping algebra as a quasi-triangular quasi-Hopf algebra, $(U(\mathcal{G})[[h]], \mathcal{R}_h, \Phi_h)$. The basic algebraic operations, multiplication, comultiplication and antipode, are undeformed, being defined on $U(\mathcal{G})[[h]]$ by the $K[[h]]$ linear extension of the standard operations on $U(\mathcal{G})$. The second result says that, modulo an equivalence relation corresponding to an equivalence of braided monoidal categories, this deformation is unique.

A non-trivial Hopf algebra deformation of $U(\mathcal{G})$, the Drinfeld-Jimbo quantum universal enveloping algebra (DJ QUE), was constructed by explicit formulae about 1985 [Dr3][J]. The existence of such a deformation together with the uniqueness theorem just mentioned proves that Drinfeld's quasi-Hopf deformation has an equivalent Hopf presentation. The equivalence is defined by an element $F_h \in (U(\mathcal{G})^{\otimes 2}[[h]]$ which transforms Φ_h to 1 and conjugates the comultiplication, see equations (17) and (26) below. From this point of view the existence F follows from the prior knowledge of the existence of a Hopf deformation.

In our approach the existence of F is proved directly. This gives an new, less ad hoc, construction of the DJ QUE and suggests an approach to the construction of other examples. We use the methods of the classical deformation theory of algebras as developed by Gerstenhaber, with Cartier coalgebra cohomology replacing Hochschild algebra cohomology. In a kind of secondary obstruction theory we shall

also use Chevalley-Eilenberg Lie algebra cohomology.

The paper is organized as follows: In §1 we give some of the basic definitions from category theory and explain the axioms for the antipode in a quasi-Hopf algebra. In §2 we review Drinfeld's cohomological proof of the existence of Φ and show how to include some additional symmetries, the significance of which have been explained in §1. Then in §3 we prove our main result, which says that, given an infinitesimal $f \in \wedge^2\mathcal{G}$, which induces the structure of a Lie bialgebra on $\mathcal{G}$, in order for there to exist a Hopf (as opposed to quasi-Hopf) quantization of $U(\mathcal{G})$, it is enough that a certain subcomplex of the Chevalley-Eilenberg complex of $\mathcal{G}_f^*$ (the induced structure on the dual of the Lie algebra $\mathcal{G}$) have zero cohomology in dimension 3. In §4 we apply this result to the DJ infinitesimal for a simple Lie algebra and give a purely cohomological proof of the existence and uniqueness up to equivalence of the DJ quantum group. Our proof is in much the same spirit as the cohomological proof of the existence of quantizations of certain Poisson brackets, [DS,Li,DL]. Having dealt in detail with the construction of the associativity constraint, Φ, in §2, in the appendix we sketch Drinfeld's cohomological proof of the existence of a pair $(\Phi_h, \mathcal{R}_h)$ satisfying the pentagon and hexagon identities. We prove that for pure imaginary deformation parameter, $\bar{h} = -h$, the elements F, R, and Φ can be chosen to be formal unitary operators on the second and third tensor powers of the regular representation of the Lie group associated to $\mathcal{G}$. Moreover, we consider some extra properties of these elements and give their interpretation in terms of additional structures on the category of representations.

§1 Preliminary categorical remarks.

Recall that a **monoidal category** is a triple $(\mathcal{C}, \otimes, \phi)$ where $\mathcal{C}$ is a category equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the tensor product, and a functorial isomorphism $\phi : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$ called associativity constraint. The

latter satisfies the pentagon identity, that is, the diagram

(5)

$$\begin{tikzcd}
((M\otimes N)\otimes P)\otimes U \arrow[r, "\phi"] \arrow[d, "\phi\otimes id"'] & (M\otimes N)\otimes(P\otimes U) \arrow[r, "\phi"] & M\otimes(N\otimes(P\otimes U)) \\
(M\otimes(N\otimes P))\otimes U \arrow[rr, "\phi"] & & M\otimes((N\otimes P)\otimes U) \arrow[u, "id\otimes\phi"']
\end{tikzcd}$$

is commutative. In addition we assume the existence of an object $\mathbf{1}$ which is a two sided identity for $\otimes$ and such that the composition

$$M\otimes N \longrightarrow (M\otimes \mathbf{1})\otimes N \xrightarrow{\phi} M\otimes(\mathbf{1}\otimes N)\longrightarrow M\otimes N \tag{6}$$

is the identity.

Let A be a Hopf algebra over a field K, of characteristic zero, and let Mod_A be the category of A modules which are finite dimensional vector spaces over K. First we define the structure of a monoidal category on Mod_A using the comultiplication. Given two representations, $\rho_M : A\to \mathrm{End}(M)$ and $\rho_N : A\to\mathrm{End}(N)$ on M and N respectively, the tensor product (over K) of vector spaces, $M\otimes N$ is naturally a representation of the tensor product of algebras

$$\rho_M\otimes\rho_N : A\otimes A\to \mathrm{End}(M)\otimes\mathrm{End}(N)\cong \mathrm{End}(M\otimes N).$$

Composition with the comultiplication

$$\rho_{M\otimes N} = (\rho_M\otimes\rho_N)\circ\Delta \tag{7}$$

defines a representation of A on $M\otimes N$. Since the comultiplication is coassociative we can take as the associativity constraint in $\mathcal{C}$ the identity morphism. The element $\mathbf{1}$ is given by the field K with A module structure coming from the augmentation of A, $\epsilon : A\to K$. For a quasi-Hopf algebra we use the same definition of the tensor product, but since the usual coassociativity condition is replaced by (1), the associativity constraint is given by the action of Φ arising from the natural $A^{\otimes 3}$ module structure on the triple tensor product of A modules. Equation (2) implies the commutativity of the pentagon (5). In order to guarantee condition (6), Φ must satisfy

$$(id\otimes\epsilon\otimes id)\Phi = 1$$

We say that a monoidal category $\mathcal{C}$ has a **rigid** monoidal structure if for each object M there is a "left dual" object M^* and a "right dual" object ${}^*M$ together with morphisms

(8)

$$\text{(a)} \quad \mathbf{1} \to M \otimes M^*, \quad \text{(b)} \quad M^* \otimes M \to \mathbf{1}, \quad \text{(c)} \quad \mathbf{1} \to {}^*M \otimes M, \quad \text{(d)} \quad M \otimes {}^*M \to \mathbf{1}.$$

For the left dual object we require that the following two compositions give identity morphisms,

$$\text{(9a)} \qquad M \to \mathbf{1} \otimes M \to (M \otimes M^*) \otimes M \xrightarrow{\phi} M \otimes (M^* \otimes M) \to M \otimes \mathbf{1} \to M,$$

$$\text{(9b)} \quad M^* \to M^* \otimes \mathbf{1} \to M^* \otimes (M \otimes M^*) \xrightarrow{\phi^{-1}} (M^* \otimes M) \otimes M^* \to \mathbf{1} \otimes M^* \to M^*,$$

and similar diagrams for the right dual.

The antipode of a (coassociative) Hopf algebra, A, is an operator, $S : A \to A$ satisfying,

$$m(S \otimes id)\Delta(a) = \epsilon(a) = m(id \otimes \Delta)S(a).$$

A rigid monoidal structure on the category Mod_A is given by defining the left dual as the vector space dual with (left module) action given by $a \cdot \lambda = \lambda \circ S(a)$. The A module morphisms, $\mathbf{1} \to M \otimes M^*$ and $M^* \otimes M \to \mathbf{1}$, in (8a,b) are defined by $k \mapsto k \cdot id$, where $M \otimes M^*$ is identified with $\mathrm{End}(M)$, and $\lambda \otimes x \mapsto \lambda(x)$ respectively. The vector space structure on the right dual is the same as the left dual but the module structure is given by $a \cdot \lambda = \lambda \circ S^{-1}(a)$.

In the case of modules over a quasi-Hopf algebra, conditions (9a,b) give two equations relating the antipode and Φ. We introduce two new elements $\alpha, \beta \in A$ coming from the definition of the morphisms in (8).

$$\mathbf{1} \to M \otimes M^* \quad \text{is given by} \quad k \to k \cdot (\rho_M(\beta) \otimes 1) \circ id$$

and

$$M^* \otimes M \to \mathbf{1} \quad \text{is given by} \quad \lambda \otimes x \to \lambda(\rho_M(\alpha)x)$$

Then the axioms for the antipode in a quasi-Hopf algebra, (equations 1.17-1.19 of [Dr1]) and equation (10) below guarantee that Mod_A is a rigid monoidal category. We use the Sweedler notation $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ and sometimes delete the summation sign for simplicity in notation.

(10a) $$\sum a_{(1)}\beta S(a_{(2)}) = \epsilon(a)\beta \quad \text{corresponding to diagram 8(a)}$$

(10b) $$\sum S(a_{(1)})\alpha a_{(2)} = \epsilon(a)\alpha \quad \text{corresponding to diagram 8(b)}$$

(10c) $$\sum \Phi_1 \beta S(\Phi_2)\alpha \Phi_3 = 1 \quad \text{corresponding to diagram 9(a)}$$

(10d) $$\sum S((\Phi^{-1})_1)\alpha(\Phi^{-1})_2 \beta S((\Phi^{-1})_3) = 1 \quad \text{corresponding to diagram 9(b).}$$

We will be interested in two supplementary conditions on Φ,

(11) $$\Phi^{321}\Phi = 1 \quad \text{and}$$

(12) $$\Phi^{321} = \Phi^S,$$

where the superscript S indicates applying the antipode to all three tensor components.

Condition (11) is a particular case (for $\mathcal{R} = 1$) of the equation

(13) $$\mathcal{R}^{21}(\Delta \otimes id)\mathcal{R} = \Phi^{321}[\mathcal{R}^{23}(id \otimes \Delta)\mathcal{R}]\Phi$$

which appears in the definition of what Drinfeld calls a coboundary structure. Condition (12) is equivalent to the compatibility of the associativity constraint with the rigid structure as expressed by the commutativity of the diagram

(14) $$\begin{array}{ccc} [M \otimes (N \otimes P)]^* & \xrightarrow{(\phi_{M,N,P})^*} & [(M \otimes N) \otimes P]^* \\ \cong \downarrow & & \cong \downarrow \\ (P^* \otimes N^*) \otimes M^* & \xrightarrow{\phi_{P^*,N^*,M^*}} & P^* \otimes (N^* \otimes M^*) \end{array}$$

where $\cong$ are compositions of the natural equivalence $(M \otimes N)^* \to N^* \otimes M^*$. We shall return to the interpretation of equations (11) and (13) after discussing the concept of equivalence of monoidal categories.

Let $(\mathcal{C}, \otimes, \phi)$ and $(\widetilde{\mathcal{C}}, \widetilde{\otimes}, \widetilde{\phi})$ be two monoidal categories, then a **monoidal functor** from $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ is given by a pair (χ, η) where $\chi : \mathcal{C} \to \widetilde{\mathcal{C}}$ is a functor and $\eta : \chi(M \otimes N) \to \chi(M) \widetilde{\otimes} \chi(N)$ is a functorial isomorphism such that the diagram

$$\begin{array}{ccccc} \chi((M \otimes N) \otimes P) & \xrightarrow{\eta} & \chi(M \otimes N) \widetilde{\otimes} \chi(P) & \xrightarrow{\eta \widetilde{\otimes} id} & (\chi(M) \widetilde{\otimes} \chi(N)) \widetilde{\otimes} \chi(P) \\ {\scriptstyle \chi(\phi)} \downarrow & & & & \downarrow {\scriptstyle \widetilde{\phi}} \\ \chi(M \otimes (N \otimes P)) & \xrightarrow{\eta} & \chi(M) \widetilde{\otimes} \chi(N \otimes P) & \xrightarrow{id \widetilde{\otimes} \eta} & \chi(M) \widetilde{\otimes} (\chi(N) \widetilde{\otimes} \chi(P)) \end{array} \tag{15}$$

is commutative and $\chi(\mathbf{1}_{\mathcal{C}}) \cong \mathbf{1}_{\mathcal{C}'}$.

For a functor between rigid monoidal categories we will require the additional conditions, $\chi(M^*) \cong \chi(M)^*$ and $\chi({}^*M) \cong^* \chi(M)$, as well as commutativity of the diagram

$$\begin{array}{ccccc} \chi((M \otimes N)^*) & \xleftarrow{\cong} & (\chi(M \otimes N))^* & \xleftarrow{(\eta_{M,N})^*} & (\chi(M) \tilde{\otimes} \chi(N))^* \\ {\scriptstyle \cong} \downarrow & & & & \downarrow {\scriptstyle \cong} \\ \chi(N^* \otimes M^*) & \xrightarrow{\eta_{N^*, M^*}} & \chi(N^*) \tilde{\otimes} \chi(M^*) & \xrightarrow{\cong} & \chi(N)^* \otimes \chi(M)^* \end{array} \tag{16}$$

and of a similar diagram for the left dual.

Given a quasi-Hopf algebra (A, m, Δ, S, Φ), let F be an element of $A^{\otimes 2}$ and define a transformation of the comutiplication on A by

$$\tilde{\Delta} = F \Delta F^{-1}. \tag{17}$$

Relative to this new comultiplication there is a new tensor product structure on Mod_A defined by

$$\rho_{M \tilde{\otimes} N} = F_{M,N} \circ \rho_{M \otimes N} \circ F^{-1}_{M,N}, \quad \text{where} \quad F_{M,N} = (\rho_M \otimes \rho_N)(F). \tag{18}$$

Then diagram (15) defines the associativity constraint, $\tilde{\phi}$, for $\tilde{\otimes}$. It is given by representing the element

$$\tilde{\Phi} = (1 \otimes F)((id \otimes \Delta)F)\Phi((\Delta \otimes id)F^{-1})(F^{-1} \otimes 1) \tag{19}$$

In other words, $\chi = id$ and

$$\eta_{M,N} = F_{M,N} : M \otimes N \longrightarrow M \tilde{\otimes} N, \tag{20}$$

define an equivalence of monoidal categories $(\mathrm{Mod}_A, \otimes, \phi)$ and $(\mathrm{Mod}_A, \tilde{\otimes}, \tilde{\phi})$. If in addition

$$(F^{21})^S F = 1 \tag{21}$$

diagram (16) and its left dual counterpart are commutative and we have an equivalence of rigid monoidal categories.

In terms of quasi-Hopf algebras, transforming Δ by (17), Φ by (19), leaving the antipode unchanged, and transforming the elements α and β, respectively, to

$$\tilde{\alpha} = \sum S(F_{1i}^{-1}) \alpha F_{2i}^{-1}, \quad \text{and} \quad \tilde{\beta} = \sum F_{1i} \beta S(F_{2i}) \tag{22}$$

defines a transformation **twisting**, which, in turn, determines an equivalence relation. If two quasi-Hopf algebras, A and A', are equivalent under twisting then the rigid monoidal categories Mod_A and $\mathrm{Mod}_{A'}$ are equivalent.

Consider $A = U(\mathcal{G})$, the universal enveloping algebra of a Lie algebra $\mathcal{G}$, and $\mathrm{Mod}_{U(\mathcal{G}),f}$, the category of finite dimensional representations. We define a deformation of the monoidal category $\mathrm{Mod}_{U(\mathcal{G}),f}$ by first extending the coefficients from the field K to the formal power series algebra $K[[h]]$. For any finite dimensional module M we define the free $K[[h]]$ module of finite rank, $M[[h]] = M \otimes_K K[[h]]$. Let $\mathrm{Mod}_{U(\mathcal{G})[[h]],f}$ be the category consisting of $U(\mathcal{G})[[h]]$ modules which are free $K[[h]]$ modules of finite rank. Define comultiplication on $U(\mathcal{G})[[h]]$ as the $K[[h]]$ linear extension of comultiplication on $U(\mathcal{G})$. Together with the tensor product of $K[[h]]$ modules this defines a monoidal structure on $\mathrm{Mod}_{U(\mathcal{G})[[h]],f}$ with associativity constraint given by the identity. Relative to this monoidal structure the imbedding defined above defines an imbedding of monoidal categories.

Drinfeld's quasi-Hopf deformation is related to a nontrivial monoidal structure on $\mathrm{Mod}_{U(\mathcal{G})[[h]],f}$. The tensor product of modules is the standard one given by (7) with the usual comultiplication but the associativity operator will be nonstandard:

(23)

where Φ_h is a $\mathcal{G}$ invariant element in $U(\mathcal{G})^{\otimes 3}[[h]]$ satisfying the pentagon identity (2).

Let $(\mathcal{C}, \otimes, \phi) = (\mathrm{Mod}_{U(\mathcal{G})[[h]],f}, \otimes, \phi_h)$ and $(\tilde{\mathcal{C}}, \tilde{\otimes}, \tilde{\phi}_h) = (\mathrm{Mod}_{U(\mathcal{G})[[h]],f}, \tilde{\otimes}, \phi_h)$ where $M\tilde{\otimes}N = N\otimes M$ and the associativity constraint is the natural one, $\tilde{\phi}_{M,N,P,h} = \phi^{-1}_{P,N,M,h}$. Then equation (11) for Φ_h is equivalent to the commutative diagram (15) where the natural transformation η is given by transposition. Equation (13) is a generalization of equation (11) in which the functor χ is the identity and the natural transformation between $M \otimes N$ and $M\tilde{\otimes}N = N \otimes M$ is given by composing transposition with the $\mathcal{R}$-matrix.

Using the standard antipode we can also define a rigid monoidal structure. In this case, the elements α and β will be chosen to be invariant, allowing us to reorder the product, and will satisfy

$$\alpha\beta = (\sum \Phi_1 S(\Phi_2)\Phi_3)^{-1}, \tag{24}$$

for example $\alpha = 1, \qquad \beta = (\sum \Phi_1 S(\Phi_2)\Phi_3)^{-1}$.

For any Lie algebra $\mathcal{G}$ defined over a field of characteristic zero, the existence of such a Φ_h satisfying (11) has been proven in [Dr1,Dr2]. In the next section we outline the proof and show that essentially the same arguments prove that Φ_h can be chosen so that it satisfies (12) as well. (In the case when $\mathcal{G}$ is simple the Φ_h is unique up to change of parameter and "twisting" as defined in (20) and (22) below. [SS])

The invariance condition on Φ_h guarantees that ϕ_h defines a natural transformation in $\mathrm{Mod}_{U(\mathcal{G})[[h]],f}$. The pentagon equation (2) for Φ_h guarantees the commutativity of (5). The triple $(\mathrm{Mod}_{U(\mathcal{G})[[h]],f}, \otimes, \phi_h)$ defines one type of deformation of the monoidal category $(\mathrm{Mod}_{U(\mathcal{G}),f}, \otimes, id)$. This is not yet the deformation which gives the quantum group since the tensor product has not been deformed. However the required deformation is given by an equivalent rigid monoidal category.

As an aside, not required in any of the subsequent proofs, but useful in understanding the situation, we relate the deformation of the category to deformation of the algebra $\mathcal{O}(G)$ of representative functions generated by the matrix entries of

ρ_M as M runs over the objects of $\mathrm{Mod}_{U(\mathcal{G}),f}$. The multiplication on $\mathcal{O}(G)$ has a simple relation to the tensor product of modules. For any pair of elements $m \in M$ and $\lambda \in M^*$ define

$$\langle u, f_{m,\lambda}\rangle = \langle \rho_M(u)m, \lambda\rangle.$$

Then the fact that multiplication in $\mathcal{O}(G)$ is dual to comultiplication in $U(\mathcal{G})$ implies immediately that

$$\begin{aligned}\langle u, f_{m,\lambda} f_{n,\mu}\rangle &= \langle \Delta(u), f_{m,\lambda} \otimes f_{n,\mu}\rangle = \sum \langle u_{(1)}, f_{m,\lambda}\rangle\langle u_{(2)}, f_{n,\mu}\rangle \\ &= \sum \langle \rho_M(u_{(1)})m, \lambda\rangle\langle \rho_N(u_{(2)})n, \mu\rangle = \langle (\rho_M \otimes \rho_N)(\Delta(u))(m \otimes n), \lambda \otimes \mu\rangle \\ &= \langle \rho_{M\otimes N}(u)(m \otimes n), \lambda \otimes \mu\rangle = \langle u, f_{m\otimes n, \lambda\otimes\mu}\rangle. \qquad (25)\end{aligned}$$

One approach to understanding the quantized function algebra is in terms of a deformation of the tensor product on the monoidal category $\mathrm{Mod}_{U(\mathcal{G}),f}$. Since any module M_h in $\mathrm{Mod}_{U(\mathcal{G})[[h]],f}$ has the form $M \otimes_K K[[h]]$, the space of matrix coefficients in $\mathrm{Mod}_{U(\mathcal{G})[[h]],f}$ can be identified with $\mathcal{O}(G)[[h]] = \mathcal{O}(G) \otimes K[[h]]$ and a tensor product on $\mathrm{Mod}_{U(\mathcal{G})[[h]],f}$ which reduces modulo h to the standard tensor product on $\mathrm{Mod}_{U(\mathcal{G}),f}$ defines a formal deformation of $\mathcal{O}(G)$. We return to this topic briefly at the end of §3.

In §3 we describe a purely cohomological proof of the existence of an element F which transforms Φ_h to 1 and conjugates the comultiplication as in (17). We prove that, under some very natural assumptions on the algebra $\mathcal{G}$, there exists an $F_h \in U(\mathcal{G})^{\otimes 2}[[h]]$ satisfying (21) and

$$(1 \otimes F_h)((id \otimes \Delta)F_h)\Phi_h((\Delta \otimes id)F_h^{-1})(F_h^{-1} \otimes 1) = 1. \qquad (26)$$

This F_h will define an equivalent rigid monoidal category to $(\mathrm{Mod}_{U(\mathcal{G})[[h]],f}, \otimes, \phi_h)$ with associativity constraint equal to the identity. At the level of quasi-Hopf algebras, we have $\Phi = 1$ so $\tilde{\alpha}\tilde{\beta} = 1$. Substituting $\tilde{S}$ defined by

$$\tilde{S}(a) = \tilde{\beta} S(a) \tilde{\alpha} \qquad (27)$$

in equations (10a) and (10b) we see that $\tilde{S}$ is an antipode satisfying the usual axioms for a Hopf algebra.

§2 Construction of a nontrivial associativity constraint.

In this section we prove a modified version of Drinfeld's theorem, [Dr1], on the existence of a nontrivial associativity constraint compatible with the undeformed tensor product. The modification involves the inclusion of additional symmetries.

Theorem 2.1. *Let $\mathcal{G}$ be a Lie algebra over a field K of characteristic zero, $U(\mathcal{G})$ its universal enveloping algebra, $K[[h]]$ be the algebra of formal power series with coefficients in K, and $U(\mathcal{G})[[h]]$ the Hopf algebra over $K[[h]]$ given by extending all operations $K[[h]]$ linearly. Let $\varphi \in \wedge^3\mathcal{G}$ be $\mathcal{G}$ invariant. Then there exists a $\mathcal{G}$ invariant formal power series $\Phi = \Phi_h \in U(\mathcal{G})^{\otimes 3}[[h]]$ solving the pentagon identity (2) and satisfying the following conditions:*

(a) $\Phi = 1 + h\varphi \mod h^2$.

(b) $\Phi^{321}\Phi = 1$.

(c) Let θ be any automorphism of $\mathcal{G}$ leaving φ invariant. If we extend θ in the natural way to $U(\mathcal{G})^{\otimes 3}[[h]]$, then Φ can be chosen satisfying, in addition to the conditions above,

$$\Phi^\theta = \Phi \tag{c}$$

(d) If we extend the antipode S to $U(\mathcal{G})^{\otimes 3}[[h]]$ in the natural way, then Φ can be chosen satisfying, in addition to (a),(b), and (c),

$$\Phi^S\Phi = 1 \tag{d}$$

Given a commuting set of θ, we can find Φ invariant under the entire set.

Proof. Rewrite the pentagon identity in the form

(28)

$$\mathrm{Pent}(\Phi) := (1\otimes\Phi)\cdot(id\otimes\Delta\otimes id)(\Phi)\cdot(\Phi\otimes 1)\cdot[(\Delta\otimes id^{\otimes 2})(\Phi)]^{-1}\cdot[(id^{\otimes 2}\otimes\Delta)(\Phi)]^{-1} = 1.$$

As usual in the theory of formal deformations we try to solve (28) recursively in powers of h. Assuming we have defined $\Phi^{(n-1)}$, an $(n-1)^{st}$ order polynomial in h satisfying (28) to order h^n, we look for an element $\varphi_n \in U(\mathcal{G})^{\otimes 3}$ such that

$\Phi^{(n)} = \Phi^{(n-1)} + \varphi_n h^n$ solves (28) to order h^{n+1}. The "obstruction" to such an extension is a cocycle in the coalgebra cohomology $U(\mathcal{G})$, whose definition we shall review quickly.

The complex has degree n component $\mathcal{C}^n = U(\mathcal{G})^{\otimes n}$, and the coboundary is induced by imbedding $\mathcal{C}^n$ in the Hochschild n cochains on $\mathcal{O}(G)$ with values in K via the pairing of $U(\mathcal{G})$ and $\mathcal{O}(G)$.

$$\langle u_1 \otimes \cdots \otimes u_n, f_1 \otimes \cdots \otimes f_n \rangle = \langle u_1, f_1 \rangle \cdots \langle u_n, f_n \rangle. \tag{29}$$

The pullback of the Hochschild coboundary to $\mathcal{C}^n$ is

(30)

$$\delta(u_1 \otimes \cdots \otimes u_n) = 1 \otimes (u_1 \otimes \cdots \otimes u_n) - \Delta(u_1) \otimes \cdots \otimes u_n + \cdots + (-1)^{n+1}(u_1 \otimes \cdots \otimes u_n) \otimes 1,$$

This coboundary operator, first introduced by Cartier [Ca], defines a cohomology controlling deformations of a coassociative coalgebra in just the same way that Hochschild cohomology controls the deformations of associative algebras. We are interested in the complex of $\mathcal{G}$ invariants, which we denote by $\tilde{\mathcal{C}}$. The cohomology is well known, see [Dr1] or [SS]:

$$H(\mathcal{C}) = \wedge \mathcal{G} \quad \text{and} \quad H(\tilde{\mathcal{C}}) = (\wedge \mathcal{G})^{\mathcal{G}}.$$

The obstructions to extending $\Phi^{(n-1)}$ are in $H^4(\tilde{\mathcal{C}}) = (\wedge^4 \mathcal{G})^{\mathcal{G}}$. Since the relevant cohomology group is in general non-zero, the obstruction cocycle for extending a truncated solution of (28) to one higher order may not cobound. Following Drinfeld, we show that condition (b) involves restricting our attention to a subcomplex with zero cohomology in dimension 4. Define an involution, τ, of $\tilde{\mathcal{C}}$ commuting with δ by

$$\tau(u_1 \otimes \cdots \otimes u_n) = (-1)^{\frac{n(n+1)}{2}} (u_n \otimes \cdots \otimes u_1). \tag{31}$$

Let $\mathcal{C}_{\tau,\pm}$ be the subcomplex consisting of the ± 1 eigenspaces τ. The complex splits as a direct sum of the two subcomplexes corresponding to the two eigenspaces. The

cohomology of each subcomplex is the corresponding eigenspace of the restriction of τ to $\wedge\mathcal{G}$. Since the action of τ on $\wedge^n\mathcal{G}$ is multiplication by $(-1)^n$,

$$H^n(\mathcal{C}_{\tau,+}) = 0 \quad \text{for n odd, and} \quad H^n(\mathcal{C}_{\tau,-}) = 0 \quad \text{for n even.}$$

This carries over to the subcomplex of $\mathcal{G}$ invariants as well.

Suppose that we have satisfied condition (b) up to order h^n, write

$$(\Phi^{(n-1)})^{321}\Phi^{(n-1)} = (\Phi^{(n-1)})^{\tau}\Phi^{(n-1)} = 1 + \eta_n h^n \mod h^{n+1}. \tag{32}$$

Assuming that $\Phi^{(n-1)}$ was $\mathcal{G}$ invariant, η_n is also. The commutativity of $\Phi^{(n-1)}$ and $(\Phi^{(n-1)})^{\tau}$ implies that $\eta_n^{\tau} = \eta_n$, so if we define

$$\Phi^{(n-1)'} = \Phi^{(n-1)} + \frac{1}{2}\eta_n h^n, \tag{33}$$

then $\Phi^{(n-1)'}$ satisfies (b) to order h^{n+1} and is $\mathcal{G}$ invariant.

Substituting in the pentagon identity we have

$$\mathrm{Pent}(\Phi^{(n-1)'}) = 1 + \xi_n h^n \tag{34}$$

for $\xi_n \in \tilde{\mathcal{C}}^4$. The fact that ξ_n is a cocycle is a rather tedious calculation given in detail in [Dr1]. Apply τ to both sides and use the fact that (b) holds to order h^{n+1}, we find that

$$1+\xi_n^{\tau} h^n = (\mathrm{Pent}(\Phi^{(n-1)'}))^{\tau} = (\mathrm{Pent}^*(\Phi^{(n-1)'})^{\tau}) = \mathrm{Pent}^*(\Phi^{(n-1)'}) = (\mathrm{Pent}(\Phi^{(n-1)'}))^{-1} = 1-\xi_n h^n \quad \mathrm{mo}$$

where

$$\mathrm{Pent}^*(\Phi) := [(id^{\otimes 2} \otimes \Delta)(\Phi)]^{-1} \cdot [(\Delta \otimes id^{\otimes 2})(\Phi)]^{-1} \cdot (\Phi \otimes 1) \cdot (id \otimes \Delta \otimes id)(\Phi) \cdot (1 \otimes \Phi).$$

Thus ξ_n is a cocycle in $\tilde{\mathcal{C}}^4_{\tau,-}$ but $H^4(\tilde{\mathcal{C}}_{\tau,-}) = 0$. Therefore ξ_n cobounds an invariant cochain $\varphi_n \in \tilde{\mathcal{C}}^3_{\tau,-}$. Define

$$\Phi^{(n)} = \Phi^{(n-1)'} + \varphi_n h^n.$$

Since $\varphi_n^{\tau} = -\varphi_n$ we have not disturbed (b) mod h^{n+1} and we have solved the pentagon identity up to order h^{n+1}.

Next we note that given θ as described in (c) there is an obvious extension, also denoted θ, to the complex $\tilde{\mathcal{C}}$ which commutes with δ and τ. Since θ is an involution, when we consider the restriction to $\tilde{\mathcal{C}}_{\tau,-}$ we have a decomposition into the direct sum of two subcomplexes, the $+1$ eigenspace and the -1 eigenspace and, as before, the cohomology also splits as a direct sum. By assumption φ is θ invariant, so we can consider solving our deformation problem in the subcomplex given by the -1 eigenspace of τ and the $+1$ eigenspace of θ. If we assume that the $\Phi^{(n-1)}$ defined above was θ invariant, then since θ is an algebra automorphism, the element η_n and the obstruction cocycle, ξ_n, are easily seen to be θ invariant and the extension to order h^{n+1} can be chosen to be θ invariant together with the other required conditions.

Finding an extension satisfying condition (d) involves a second preliminary correction. Assume that $\Phi^{(n-1)'}$ has been defined as in (33) so that condition (b) is satisfied to order h^{n+1} and that condition (d) has been satisfied to order h^n. Define χ_n by

$$(\Phi^{(n-1)'})^S\Phi^{(n-1)'} = 1 + \chi_n h^n \mod h^{n+1}. \tag{35}$$

Then

$$\begin{aligned} 1 + \chi_n^{321} h^n &= (\Phi^{(n-1)'321})^S \Phi^{(n-1)'321} \\ &= ((\Phi^{(n-1)'})^{-1})^S (\Phi^{(n-1)'})^{-1} \\ &= (\Phi^{(n-1)'}\Phi^{(n-1)'S})^{-1} \\ &= 1 - \chi_n h^n \mod h^{n+1}. \end{aligned}$$

Thus $(\chi_n)^{321} = -\chi_n$ and, as before, $\chi_n^S = \chi_n$. Therefore if we set $\Phi^{(n-1)''} = \Phi^{(n-1)'} + \frac{1}{2}\chi_n h^n$ we satisfy do not disturb condition (b) and satisfy condition (d) to one higher order. Thus the preliminary corrections for conditions (b) and (d) can be done simultaneously. If both (b) and (d) are true to order h^{n+1}, then

$$1 + (\zeta_n^\omega + \zeta_n)h^n = (1 + \zeta_n h^n)^\omega (1 + \zeta_n h^n) = \mathrm{Pent}^*(\Phi^{(n-1)'\omega} \quad \mathrm{Pent}(\Phi^{(n-1)'}) = 1 \mod h^{n+1}$$

where ω is either S or τ. Therefore the obstruction cocycle ξ_n is in the subcomplex given by the intersection of the -1 eigenspaces of τ and S. As before ξ_n cobounds a cochain φ_n of the same type. The element $\Phi^{(n)} = \Phi^{(n-1)'} + \varphi_n h^n$ satisfies all the required conditions to order $h^{(n+1)}$. $\square$

Note that if we have a commuting set of automorphisms, θ_i, then all the decompositions into eigenspaces of τ, θ_i and S are compatible and we can solve in the subcomplex consisting of -1 eigenvectors of τ, $+1$ eigenvectors of θ_i and $+1$ eigenvectors of S.

In particular, for $\mathcal{G}$ a simple Lie algebra we can let θ be the Cartan involution corresponding to a choice of Cartan subalgebra and system of positive roots and, this proves the existence of a $\mathcal{G}$ invariant Φ satisfying (11) and (12).

Several remarks are in order before closing this section. First of all, in the next section we want to consider another Φ which is given by replacing the deformation parameter h with h^2. will be necessary in order to construct the equivalence F. Henceforth, when referring to Φ we intend this form. Second, we note that the invariance of Φ implies the invariance of $\gamma = \sum \Phi_1 S(\Phi_2)\Phi_3 \in U(\mathcal{G})[[h]]$. If we chose α and β invariant and $\alpha\beta = \gamma^{-1}$ and let S be the $K[[h]]$ linear extension of the standard antipode for $U(\mathcal{G})$, then equations (10a-d) are satisfied, making $U(\mathcal{G})[[h]]$ a quasi-Hopf algebra. For example, we can chose $\alpha = 1$ and $\beta = \gamma^{-1}$.

Finally, the construction given above does not involve the braiding. Drinfeld [Dr2] has given a cohomological proof of the existence of a pair $(\mathcal{R}, \Phi)$ satisfying (3), (4ab) and (11) thus defining a quasitriangular quasi-Hopf structure on $U(\mathcal{G})[[h]]$. This proof, which is much more delicate, is sketched in the appendix where we also discuss the modifications necessary to deal with additional symmetries.

§3 Construction of a quantized universal enveloping algebra.

In this section we study sufficient conditions for the existence of a solution to equation (26). These conditions are satisfied, in particular, in the case of the Drinfeld-Jimbo infinitesimal for $\mathcal{G}$ a simple Lie algebra. Twisting the Φ_h constructed above by $F = F_h \in U(\mathcal{G})^{\otimes 2}[[h]]$ and transforming the antipode to $\tilde{S}$ as

given by (27), defines a quantized universal enveloping algebra. Note that this presentation is not the standard one since the multiplication is undeformed.

Our proof uses a combination of coalgebra cohomology and Chevalley-Eilenberg cohomology for the Lie algebra structure on $\mathcal{G}^*$ associated to the leading nonconstant term of F. Recall that we are assuming that Φ has the form

$$\text{(36a)} \quad \Phi = 1 + \varphi h^2 + \sum_{m>1} \varphi_{2m} h^{2m}, \quad \text{with} \quad \varphi \in (\wedge^3 \mathcal{G})^{\mathcal{G}} \quad \text{and} \quad \varphi_{2m} \in (U(\mathcal{G})^{\otimes 3})^{\mathcal{G}},$$

and we set

$$\text{(36b)} \qquad F = 1 + fh + \sum_{n>1} f_n h^n, \quad \text{with} \quad f, f_n \in U(\mathcal{G})^{\otimes 2}.$$

The element f will be called the infinitesimal of F.

It will be convenient to reformulate equation (26) in the form

$$\text{(37)} \qquad B(F_h, \Phi_h) = (1 \otimes F_h)((id \otimes \Delta)F_h)\Phi_h - (F_h \otimes 1)(\Delta \otimes id)F_h = 0.$$

Consider the h and h^2 terms in (37):

(38)

$$B(1+fh+f_2h^2, 1+\varphi h^2) = (\delta f)h + (f^{23}(f^{12}+f^{13}) - f^{12}(f^{13}+f^{23}) + \delta f_2 + \varphi)h^2 \mod h^3,$$

where δ is the Cartier coboundary, (30).

Two remarks:

(1) The vanishing of the first term implies that f must be a δ cocycle.

(2) If Φ were expanded in powers of h, instead of powers of h^2, and φ was the coefficient of h then (37) would require $\delta f + \varphi = 0$ which implies $\varphi = 0$ since $\wedge \mathcal{G}$ is transversal to the subspace of coboundaries.

If the multiplicative group, $1 + hU(\mathcal{G})[[h]]$ acts on the left on $U(\mathcal{G})^{\otimes 2}[[h]]$ by

$$\text{(39)} \qquad u \bullet F = (u \otimes u)F\Delta(u^{-1}),$$

then

$$\text{(40)} \qquad B(u \bullet F, \Phi) = (u \otimes u \otimes u)B(F, \Phi)(\Delta \otimes id)(\Delta(u^{-1}))$$

Therefore this action carries solutions of (37) into solutions and we call two solutions equivalent if they are in the same orbit. If $F' = u \bullet F$ for $u = 1 + u_1 h \mod h^2$, $u_1 \in U(\mathcal{G})$, then the infinitesimals are related by $f' = f + \delta u_1$. Since the δ cohomology in dimension 2 is $\wedge^2\mathcal{G}$, in any equivalence class there is a representative with infinitesimal

$$f \in \wedge^2\mathcal{G}, \tag{41}$$

so making this assumption in (36b) involves no loss of generality.

Now consider the requirement on the pair f, f_2 implied by (38).

$$f^{23}(f^{12} + f^{13}) - f^{12}(f^{13} + f^{23}) \tag{42}$$

must belong to the cohomology class of $-\varphi$. The fact that (42) is a cocycle follows from the following lemma.

Lemma 3.1. *Let $F^{(n-1)}$ be an $(n-1)^{st}$ order polynomial in h of the form (36b) with infinitesimal (41) and suppose that $F^{(n-1)}$ satisfies (37) to order h^n. Define the obstruction cochain, ξ_n, by*

$$B(F^{(n-1)}, \Phi) = \xi_n h^n \quad mod \quad h^{n+1}. \tag{43}$$

(1) *The cochain ξ_n is a cocycle.*
(2) *If $\delta f_n = \xi_n$ then $F^{(n)} = F^{(n-1)} + f_n h^n$ defines an extension satisfying (37) to order h^{n+1}.*

Proof. Multiplying (43) on the right by $[\Delta \otimes id(F^{(n-1)})(F^{(n-1)} \otimes 1)]^{-1}$ and recalling the form of B given in (37),

$$\begin{aligned}\tilde{\Phi} &= (1 \otimes F^{(n-1)})((id \otimes \Delta)F^{(n-1)})\Phi((\Delta \otimes id)(F^{(n-1)})^{-1})((F^{(n-1)})^{-1} \otimes 1) \\ &= 1 + \xi_n h^n \quad \mod \quad h^{n+1}.\end{aligned}$$

A straightforward calculation shows that if Φ satisfies the pentagon identity relative

The congruence $\Delta = \tilde{\Delta} \mod h$, implies that the operator, $\tilde{\delta}$, defined by (30) with $\tilde{\Delta}$ replacing Δ is congruent mod h to δ, therefore

$$1 = \widetilde{\mathrm{Pent}}(1 + \xi_n h^n)) = 1 + \tilde{\delta}\xi_n h^n = 1 + \delta\xi_n h^n \mod h^{n+1}$$

so ξ_n is a δ cocycle.

The second statement follows from the equation

$$B(F^{(n-1)} + f_n h^n, \Phi) = B(F^{(n-1)}, \Phi) + (\delta f_n)h^n \mod h^{n+1}. \quad \square$$

Since $H^n(\mathcal{C}) = \wedge^n \mathcal{G}$, every n-cocycle is cohomologous to an element of $\wedge^n \mathcal{G}$ which we take as the canonical representative of its cohomology class. Furthermore antisymmetrization annihilates coboundaries and projects a cocycle onto the canonical representative. Antisymmetrizing (42) gives $-\frac{2}{3}$ times the Yang-Baxter expression $YB(f)$,

$$YB(f) := [f^{12}, f^{13} + f^{23}] + [f^{13}, f^{23}] = -[f^{23}, f^{12} + f^{13}] + [f^{12}, f^{13}]. \tag{44}$$

Lemma 3.2. *The necessary and sufficient condition for the existence of a solution of (37) up to order h^3, is that the infinitesimal f satisfy*

$$YB(f) = \frac{3}{2}\varphi. \tag{45}$$

Proof. Assuming that f satisfies (45), the antisymmetrization of (42) is $-\varphi$ as required for the obstruction to be a coboundary. Thus there exists an f_2 such that $F^{(2)} = 1 + fh + f_2 h^2$ satisfies (37) to order h^3. $\square$

In order to continue the process we need to consider conditions on F which guarantee that the higher obstructions cocycles cobound. The strategy suggested by Drinfeld's proof of the existence of Φ is to impose conditions on F that force the obstructions to lie in a subcomplex with trivial 3-cohomology.

In the case of a general Lie algebra, there are two obvious symmetries that we can impose on F. The first one, relating to the antipode, was introduced in equation (21). The second condition on F is

$$F^{21} = F, \quad \text{or in components} \quad f_n^{21} = (-1)^n f_n \tag{46}$$

This can be interpreted in terms of the $*_h$ product on the function algebra as saying that the commutator and anticommutator are, respectively, expansions in odd and even powers of h.

The following lemma shows that in solving equation (37) recursively using obstruction theory we can reduce to a subcomplex of invariants with respective the symmetries (θ, S) of Φ.

Lemma 3.3. *Let $\Phi = \Phi_h$ be a solution to the pentagon identity satisfying (11),(12) and $\Phi_{-h} = \Phi_h$. Let*

$$F' = 1 + fh + \sum_{k=1}^{n-1} f_k h^k$$

be a solution of (37) satisfying (46) and (21), all this modulo h^n, for $n \geq 3$.

(a) Define η, F and ξ by

$$(F'^{21})^S F' = 1 + \eta h^n \quad mod \quad h^{n+1}$$
$$F = F' + \frac{1}{2}\eta h^n$$
$$B(F, \Phi) = \xi h^n \quad mod \quad h^{n+1}.$$

Then F satisfies (21) and (46) mod h^{n+1} and $\xi^\tau = \xi^S = (-1)^{n+1}\xi$, where τ is defined by (31). For n odd the obstruction cocycle cobounds and we can extend $F^{(n-1)}$ to a solution to (37) to order h^{n+1} which still satisfies (21) and (46) to the same order.

(b) Let θ be an involutive automorphism of $\mathcal{G}$ and use the same symbol to denote its extension to the complex $\mathcal{C}$. Assume that a solution, Φ, to the pentagon equation has been constructed such that $\Phi^\theta = \Phi$. If $F^\theta = F$ then $\xi^\theta = \xi$. If $F_h^\theta = F_{-h}$ then $\xi^\theta = (-1)^n \xi$.

Proof. Define $\hat{F}$ ... F ... Using successively the symmetry (46) on F together

with the identities $\hat{\Phi} = \Phi, \quad \Phi\Phi^{321} = 1$, $\Phi = 1 \bmod h^2$ we get, modulo h^{n+1},

$$\begin{aligned}
\xi_n^{321} h^n &= B(F,\Phi)^{321} = F^{21}(\Delta \otimes id)(F^{21})\Phi^{321} - F^{32}(id \otimes \Delta)(F^{21}) \\
&= \hat{F}^{12}(\Delta \otimes id)\hat{F}\Phi^{321} - \hat{F}^{23}(id \otimes \Delta)(\hat{F}) \\
&= -B(\hat{F},\Phi)\Phi^{321} = -B(\hat{F},\Phi) = -\xi_n(-h)^n \\
&= (-1)^{n+1}\xi_n h^n.
\end{aligned}$$

This shows that the obstruction cocycle at odd order lies in the acyclic subcomplex $\mathcal{C}_{\tau,+}$ and so it is a coboundary. To prove the second identity on ξ in part (a) we use the additional facts that $F = 1 \bmod h$, (21) has been solved modulo h^{n+1} and $(\Phi^{321})^S = \Phi$. Again, computing modulo h^{n+1},

$$\begin{aligned}
(\xi_n^{321})^S h^n &= (B(F,\Phi)^{321})^S = (\Phi^{321})^S(\Delta \otimes id)(F^{21})^S(F^{21})^S - (id \otimes \Delta)(F^{21})^S(F^{32})^S \\
&= \Phi(\Delta \otimes id)(F^{-1})(F^{12})^{-1} - (id \otimes \Delta)(F^{-1})(F^{23})^{-1} \\
&= (id \otimes \Delta)(F^{-1})(F^{23})^{-1}B(F,\Phi)(\Delta \otimes id)(F^{-1})(F^{12})^{-1} \\
&= (id \otimes \Delta)(F^{-1})(F^{23})^{-1}(\xi_n h^n)(\Delta \otimes id)(F^{-1})(F^{12})^{-1} = \xi_n h^n.
\end{aligned}$$

Condition (b) follows from

$$\xi^\theta h^n = B(F,\Phi)^\theta = B(F^\theta,\Phi^\theta) = B(F,\Phi) = \xi h^n \quad \mathrm{mod} \quad h^{n+1}.$$

The second part is proved in the same way

$$\xi(-h)^n = B(\hat{F},\Phi) = B(F^\theta,\Phi^\theta) = B(F,\Phi)^\theta = \xi^\theta h^n \quad \mathrm{mod} \quad h^{n+1}.$$

□

Remark 3.1 We shall use the concept of a bialgebra action to consider invariance with respect to the Cartan subalgebra, $\mathcal{H}$, of $\mathcal{G}$. If $(\mathcal{U},\Delta)$ and $(\mathcal{P},\bar{\Delta})$ are two bialgebras, then a bialgebra action of $\mathcal{P}$ on $\mathcal{U}$ is a $\mathcal{P}$ module structure on $\mathcal{U}$ satisfying the two conditions

$$p \cdot (uv) = \sum (p_{(1)} \cdot u)(p_{(2)} \cdot v) \quad \text{and} \quad \Delta(p \cdot u) = p \cdot \Delta(u)$$

In item (b) of Lemma 3.1 we have extended the action to $\mathcal{C}^n = \mathcal{U}^{\otimes n}$ by

$$p \cdot (u_1 \otimes \cdots \otimes u_n) = \sum (p_{(1)} \cdot u_1) \otimes \cdots \otimes (p_{(n)} \cdot u_n),$$

where $\sum p_{(1)} \otimes \cdots \otimes p_{(n)}$ represents the n^{th} iteration of $\bar{\Delta}$. We say that an element of $\Psi \in \mathcal{C}$ is invariant under $\mathcal{P}$ if $p \cdot \Psi = \epsilon(p)\Psi$ for all $p \in \mathcal{P}$. The first of the two properties implies that the product of invariants is an invariant, thus if Φ and F are $\mathcal{P}$ invariant then so is ξ. The second property implies that the coboundary commutes with the action so the invariants form a subcomplex. If θ is the Cartan involution, the actions of $\mathcal{P}, \theta, S$ are all compatible and we can add the condition of $\mathcal{P}$ invariance in Lemma 3.3:

For the Drinfeld-Jimbo infinitesimal, the relevant bialgebra is $\mathcal{P} = U(\mathcal{H})$ and the relevant involution is the Cartan involution. The subcomplex of invariants is a direct summand of the total complex. The even obstructions are invariant under θ and $\mathcal{H}$. Unfortunately, this still does not make the obstructions cohomology classes zero.

The next idea is to cancel the nonvanishing obstruction cohomology class using the Chevalley-Eilenberg cohomology of the Lie algebra structure on $\mathcal{G}^*_f$ defined by the infinitesimal f. The fundamental proposition is once again due to Drinfeld.

Proposition 3.1 (Drinfeld[Dr3]**).** *For $f \in \wedge^2\mathcal{G}$ the $\mathcal{G}$ invariance of $YB(f)$ is equivalent to the Jacobi identity for the bracket $[\lambda, \mu]$ on $\mathcal{G}^*$ defined by*

$$\langle [X \otimes 1 + 1 \otimes X, f], \lambda \wedge \mu \rangle = \langle X, [\lambda, \mu] \rangle.$$

Now suppose that we are given F defined to order h^{2m-1} with obstruction cocycle ξ. If we change F by adding $\chi \in \wedge^2\mathcal{G}$

$$F' = F + \chi h^{2m-1},$$

the new obstruction cocycle is

$$B(F', \Phi) = (\xi + b(f, \chi))h^{2m}$$

where

$$b(f,\chi) = f^{23}(\chi^{12}+\chi^{13}) + \chi^{23}(f^{12}+f^{13}) - f^{12}(\chi^{13}+\chi^{23}) - \chi^{12}(f^{13}+f^{23}).$$

Projecting onto cohomology by antisymmetrization, we get

$$\mathrm{Alt}(\xi + b(f,\chi)) = \mathrm{Alt}(\xi) - \frac{2}{3}\widetilde{YB}(f,\chi) \tag{47}$$

where $\widetilde{YB}$ is the polarization of (42):

$$[f^{12},\chi^{13}+\chi^{23}] + [f^{13},\chi^{23}] + [\chi^{12}, f^{13}+f^{23}] + [\chi^{13}, f^{23}]. \tag{48}$$

Thus we need to choose χ so that (47) is zero. Formula (48) is a particular example of the **Schouten bracket**

$$\begin{gathered} [[\cdot,\cdot]] : \wedge^k\mathcal{G} \otimes \wedge^l\mathcal{G} \longrightarrow \wedge^{k+l-1}\mathcal{G} \\ [[X_1\wedge\cdots\wedge X_k, Y_1\wedge\cdots\wedge Y_l]] = \sum(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\cdots\hat{X}_i\cdots\hat{Y}_j\cdots\wedge Y_l. \end{gathered} \tag{49}$$

Thus (47) is zero if and only if

$$\frac{3}{2}\,\mathrm{Alt}\,\xi = [[f,\chi]]. \tag{50}$$

We remind the reader of the well known fact.

Proposition 3.2. *If we identify $\wedge^n\mathcal{G}$ with a skew symmetric multilinear maps on $\mathcal{G}^*$, then the Chevalley-Eilenberg coboundary d_{CE} for the cohomology of the Lie algebra $\mathcal{G}^*_f$ with coefficients in the ground field is given up to a normalizing factor by the Schouten bracket, $d_{CE} = [[f,\cdot]$.*

Proof. For $X \in \mathcal{G}$ and $\lambda,\mu \in \mathcal{G}^*$, by definition of the bracket on $\mathcal{G}^*$ we have

$$\langle [[X,f]], \lambda\wedge\mu\rangle = \langle [X\otimes 1 + 1\otimes X, f], \lambda\wedge\mu\rangle = \langle X, [\lambda,\mu]\rangle = \langle d_{CE}X, \lambda\wedge\mu\rangle$$

Since both $[[\cdot,f]]$ and d_{CE} are derivations of the exterior algebra, $\wedge\mathcal{G}$, and they agree on the generators, they are equal. $\square$

If $\mathrm{Alt}\,\xi$ is a coboundary in the Chevalley-Eilenberg cohomology then we can adjust the extension at order h^{2m-1} by adding χh^{2m-1} and cancel the obstruction in δ cohomology at order h^{2m} . Thus we have a kind of secondary obstruction theory. The next lemma makes this more precise.

Lemma 3.4. *Let ξ be the obstruction cocycle for $F^{(n-1)}$ as defined in (43), then $d_{CE}\operatorname{Alt}(\xi) = 0$. If $d_{CE}\,\chi = \frac{3}{2}\operatorname{Alt}(\xi)$ then the obstruction cocycle for $F' = F + \chi h^{n-1}$ is a δ coboundary.*

Proof. As in the proof of Lemma 3.1 we use the fact that $1 + \xi h^n$ satisfies the $\tilde{\Delta}$ pentagon identity. Let $\tilde{\delta}$ be given by formula (30) with $\tilde{\Delta}$ replacing Δ and $\eta = \operatorname{Alt}\xi$. Then

$$\tilde{\Delta}(u) = \Delta(u) + \Delta_1(u)h = \Delta(u) + [f, \Delta(u)]h, \quad \mod \quad h^2,$$

$$\text{Since} \quad 1 = \widetilde{\operatorname{Pent}}(1 + \xi h^n) = 1 + \tilde{\delta}\xi h^n \quad \mod \quad h^{n+2},$$

$$\begin{aligned}
\text{we have} \quad 0 &= \operatorname{Alt}\tilde{\delta}\xi = \operatorname{Alt}(1 \otimes \xi - (\tilde{\Delta} \otimes 1 \otimes 1)\xi + (1 \otimes \tilde{\Delta} \otimes 1)\xi - (1 \otimes 1 \otimes \tilde{\Delta})\xi + \xi \otimes 1 \\
&= (-(\tilde{\Delta}_1 \otimes 1 \otimes 1)\eta + (1 \otimes \tilde{\Delta}_1 \otimes 1)\eta - (1 \otimes 1 \otimes \tilde{\Delta}_1)\eta)h \\
&= (-[f^{12}, \Delta \otimes 1 \otimes 1(\eta)] + [f^{23}, 1 \otimes \Delta \otimes 1(\eta)] - [f^{34}, 1 \otimes 1 \otimes \Delta(\eta)])h \\
&= -[[f, \eta]]h = -d_{CE}(\eta)h \quad \mod \quad h^2.
\end{aligned}$$

The second part of the lemma follows immediately from (50). In [Dr3], Drinfeld pointed out the relevance of the Chevalley-Eilenberg cohomology of $\mathcal{G}_f^*$ to the study of bialgebra deformations. This was developed further by LeComte and Roger [LR]. Our approach is slightly different since we use both the Chevalley-Eilenberg cohomology and the Cartier cohomology.

Theorem 3.1. *Given $\mathcal{G}$ be a Lie algebra over a field of characteristic zero and an $f \in \wedge^2\mathcal{G}$ such that $[[f, f]]$ is $\mathcal{G}$ invariant. Let $\mathcal{G}_f^*$ be the Lie algebra structure induced by f on $\mathcal{G}^*$. Suppose we are in the situation of Remark 3.1 and a bialgebra $\mathcal{P}$ acts on $U(\mathcal{G})$ and that it preserves $\mathcal{G}$. Then both $\mathcal{P}$ and θ act on the Chevalley-Eilenberg cohomology of $\mathcal{G}_f^*$. Assume that f is $\mathcal{P}$ and θ invariant. If, in $H^3_{CE}(\mathcal{G}_f^*)$, the subspace, $H^3_{CE}(\mathcal{G}_f^*)'$ of invariants of both actions is zero, then there exists a quantized universal enveloping algebra having undeformed multiplication and deformed comultiplication with infinitesimal induced by the commutator $[f, \Delta]$.*

Proof. Define $\varphi = \frac{2}{3}[[f, f]]$. Then applying Theorem 2.1 we construct Φ satisfying (11), (12) and $\Phi^\theta = \Phi$. If there exists a solution F_n to (26) we can define Δ

$F_h \Delta F_h^{-1}$. Lemma 3.2 says that we can find such a solution modulo h^3. As noted earlier, Lemma 3.3 implies that the only obstructions lie in the subcomplex of θ and $\mathcal{P}$ invariants and Lemma 3.4 shows that it is enough to consider the invariants in the Chevalley-Eilenberg cohomology. If this space of invariants is zero, there are no obstructions to the recursive solution of (26). $\square$

Associated to any deformation of comultiplication in $U(\mathcal{G})$ there is a dual deformation of the multiplication in $\mathcal{O}(G)$, which is expressed by

$$f_{m,\lambda} *_h f_{n,\mu} = f_{m,\lambda} \otimes f_{n,\mu} \circ F_h \Delta F_h^{-1}. \tag{51}$$

We shall check the compatibility with the formula

$$f_{m,\lambda} *_h f_{n,\mu} = f_{m \tilde{\otimes} n, \mu \tilde{\otimes} \lambda}. \tag{52}$$

By the form of the equivalence, equation (20), $m \tilde{\otimes} n = (F^{-1})_{(1)}(m) \otimes (F^{-1})_{(2)}(n)$ and $\mu \tilde{\otimes} \lambda = (F^{-1})_{(1)}(\mu) \otimes (F^{-1})_{(2)}(\lambda) = \mu \circ S((F^{-1})_{(1)}) \otimes \lambda \circ S((F^{-1}))_{(2)} = \mu \circ F_{(2)} \otimes \lambda \circ F_{(1)}$, where the last equation follows from (21). These identities, when substituted in the right side of (51) give (52). The Hopf algebra $\mathcal{O}(G)[[h]]$ with this product $*_h$, undeformed comultiplication, and dual antipode, $\tilde{S}'$, is the function algebra of a quantum group in a nonstandard presentation.

§4 Drinfeld-Jimbo quantization of $U(\mathcal{G})$.

To construct the Drinfeld-Jimbo QUE algebra for a semisimple Lie algebra $\mathcal{G}$, choose a Cartan subalgebra and a system of positive roots, Π. Let

$$f = \sum_{\alpha \in \Pi} X_\alpha \wedge X_{-\alpha} \tag{51}$$

be the Drinfeld-Jimbo classical R-matrix. It is $\mathcal{H}$ invariant and skew invariant under the Cartan involution. We shall apply Theorem 3.1 to the case when $\mathcal{P} = U(\mathcal{H})$ and θ is the Cartan involution.

If we use the Killing form on $\mathcal{G}$ to define an isomorphism between $\mathcal{G}$ to $\mathcal{G}^*$ then the Lie algebra structure $\mathcal{G}^*_f$ can be transferred to a new Lie algebra structure, denoted $\mathcal{G}_{[illegible]}$ on $\mathcal{G}$.

Lemma 4.1. *The space of invariants under $\mathcal{H}$ and θ, $H^3_{CE}(\mathcal{G}_f)'$, is trivial.*

Proof. The Lie algebra, $\mathcal{G}_f$, can be described quite simply in two steps. Let $\mathcal{N}_\pm$ be the two subalgebras of $\mathcal{G}$ consisting of the sum of the positive root spaces and the sum of the negative root spaces, respectively. First form the Lie algebra sum $\mathcal{N}_+ \oplus \mathcal{N}_-$ then take the semidirect product with $\mathcal{H}$ where $\mathcal{H}$ acts on $\mathcal{N}_+$ by the standard bracket and on $\mathcal{N}_-$ by the negative of the standard bracket.

In general, given an abelian subalgebra acting semisimply under the adjoint representation, the non-trivial cohomology lies entirely in the subcomplex of zero weight. For the Lie algebra $\mathcal{G}_f$ the relevant subcomplex of $\wedge\mathcal{G}$ is just $\wedge\mathcal{H}$. Since f is $\mathcal{H}$ invariant the Schouten bracket $[[f,\cdot]]$ restricts to the zero operator on the subcomplex and therefore the cohomology is $H^n_{CE}(\mathcal{G}^*_f) = \wedge^n\mathcal{H}$. The Cartan involution restricted to $\mathcal{H}$ is -1 so the subspace of invariants in any odd exterior power is zero. $\square$

We can now state our main result which follows immediately from Theorem 3.1.

Theorem 4.1. *Let $\mathcal{G}$ be a semisimple Lie algebra over a field of characteristic zero and f the Drinfeld-Jimbo classical R-matrix as defined in (51). Then there exists an $F = F_h \in U(\mathcal{G})^{\otimes 2}[[h]]$ such that*

1. *F transforms the associativity constraint to the identity (equation (26)).*
2. *F is invariant under $\mathcal{H}$ and any automorphism group of $\mathcal{G}$ under which f is invariant.*
3. *$F^\theta_h = F^{21}_h = F_{-h}$, where θ is the Cartan involution.*
4. *$F^S F^{21} = 1$, where S is the antipode.*

Corollary 4.1. *Let $\mathcal{G}$ and F be as in Theorem 4.1. Define a quasi-bialgebra deformation of $U(\mathcal{G})$ by leaving the multiplication undeformed and deforming the comultiplication by twisting by F_h*

$$\Delta_h(x) = F_h\Delta(x)F_h^{-1}$$

The resulting deformation has the following properties:

(1) *The deformed comultiplication is coassociative.*

(2) *The restriction of Δ_h to $U(\mathcal{H})$ is the standard, undeformed, so $U(\mathcal{H})$ is undeformed. comultiplication.*

(3) *θ is a coalgebra antiautomorphism relative to Δ_h.*

(4) *The undeformed antipode S is a coalgebra antiautomorphism relative to Δ_h.*

(5) *Let $F = \sum F_{1i} \otimes F_{2i}$ and $w = \sum F_{2i} S(F_{1i})$. Then*

$$\tilde{S}(u) = w^{-1} S(u) w$$

defines an antipode relative to which the deformation is a Hopf algebra.

Proof. Each item, except the last, follows from the corresponding item in Theorem 4.1. The last statement follows from the explanation of the twisting transformation and equation (27) with $\alpha = 1$. $\square$

Regarding the uniqueness of the solution, F, with a given infinitesimal, f, we have the following proposition.

Proposition 4.1. *Let $\mathcal{G}$ be a Lie algebra of a field of characterisitic zero, and $\Phi \in U(\mathcal{G})^{\otimes 3}[[h]]$ a solution to the pentagon identity. Given two solutions, F_h and F_h', to equations (21) and (26) both with initial term $1 \otimes 1$ and the same infinitesimal $f \in \wedge^2 \mathcal{G}$, then there exists a $u_h \in U(\mathcal{G})[[h]]$ such that*

$$F_h = (u_h \otimes u_h) F_h' \Delta(u_h^{-1}), \quad \textit{and} \quad u_h S(u_h) = 1.$$

Proof. We shall prove, as usual, that if F and F' agree to order h^n then there exists a $u = 1 + u_n h^n$ such that mod h^{n+1} F equals the transform of F', $(u \otimes u) F' \Delta(u^{-1}) = F' + \delta(u_n) h^n$. The facts that F and F' both satisfy (26) and that they agree to order h^n imply that the difference of the coefficients of h^n is a δ cocycle, $\delta(f_n - f_n') = 0$. Therefore, there exists a v such that $f_n - f_n' = \delta v + w$, where $w \in \wedge^2 \mathcal{G}$. However (21) implies that $f_n + f_n^S = f_n' + (f_n')^S$. Thus $(f_n - f_n') = -(f_n - f_n')^S$. Now $w = S(w)$ and $(\delta v)^S = \delta(S(v))$, so we have $f_n - f_n' = \frac{1}{2}((f_n - f_n') - (f_n - f_n')^S) = \delta(\frac{1}{2}(v - S(v)))$.

Set $u_n = v - S(v)$

For an arbitrary pair, F_h and F'_h, the h expansion of u_h is given by an infinite product

$Lim_{n\to\infty}(1+u_nh^n)\cdots(1+u_2h^2)$. Next consider the condition $S(u_h)u_h = 1$. Suppose that we have defined u' which is the product of the first n terms, so that, modulo h^{n+1}, F and F' agree and that $S(u')u' = 1+\xi_nh^n$. The fact that both $F = (u'\otimes u')F'(\Delta(u')^{-1})$ and F' satisfy (21) modulo h^{n+1} implies that (modulo h^{n+1})

$$\Delta(S(u')^{-1})(F')^S(S(u')\otimes S(u'))(u'\otimes u')F'(\Delta(u')^{-1}) = 1.$$

Substituting in $(1+\xi_nh^n)u'^{-1}$ for $S(u')$ and expanding the product, using (21) and $F = F' = 1$ modulo h, we see that $\xi_n \in U(\mathcal{G})$ is a δ cocycle, i.e., primitive relative to Δ, so it is an element of $\mathcal{G}$. From the definition it also follows that ξ is S invariant, so, in fact, it is zero and no correction is necessary. $\square$

In Theorem 4.1 we allow f to be any linear combination of the Drinfeld-Jimbo infinitesimals for the simple factors. This possibility is important for applications to homogeneous spaces, in particular the symmetric space $(G\times G)/G$. When $\mathcal{G}$ is simple, the deformation described in Theorem 4.1 is unique up to inner automorphism and change of parameter, as was noted in [Dr3]. For a complete proof see [SS].

In this way it is possible to quantize some of the classical R-matrices classified by Belavin Drinfeld. These examples will be discussed in a future paper.

Another interesting application of Theorem 3.1 is the quantization of an arbitrary infinitesimal f with invariant Schouten bracket $[[f,f]]$ for any three dimensional Lie algebra. In this case $\wedge^3\mathcal{G}$ is one dimensional, if $[[f,\chi]] \neq 0$ for some $\chi \in \wedge^2\mathcal{G}$ then $H^3_{CE}(\mathcal{G}^*_f) = 0$. On the other hand, suppose $[[f,\chi]] = 0$ for all χ. We can choose a basis $\{X,Y,Z\}$ such that $f = X\wedge Y$. The condition that $[[f,f]] = 0$ implies that $[X,Y] \in \mathrm{span}\{X,Y\}$. The conditions $[[f,Y\wedge Z]] = 0$ and $[[f,X\wedge Z]] = 0$ imply that $[X,Y] \in$ span $\{Y,Z\}$ and $[X,Y] \in$ span $\{X,Z\}$ respectively. All the conditions together imply $[X,Y] = 0$. In this case $F = e^{hf}$ satisfies (26) with $\Phi = 1$ and gives the desired quantization.

§Appendix. Auxiliary conditions on the R-matrix.

In this appendix we return to the braiding and consider the problem of proving the existence of a quasi-triangular quasi-Hopf deformation $(U(\mathcal{G})[[h]], \mathcal{R}_h, \Phi_h)$ which includes auxiliary conditions on $\mathcal{R}$ corresponding to the conditions on Φ appearing in Theorem 2.1. The correspondence is shown in the table given in the following proposition, where S is the antipode, $(\hat{\mathcal{R}})_h = \mathcal{R}_{-h}$ and similarly for Φ.

Proposition. *For any $\mathcal{G}$ invariant symmetric invariant element $t \in \mathcal{G} \otimes \mathcal{G}$ and automorphism θ preserving t there exists a pair of $\mathcal{G}$ invariant elements $\Phi \in U(\mathcal{G})^{\otimes 3}[[h]]$ and $\mathcal{R} \in U(\mathcal{G})^{\otimes 2}[[h]]$, where $\mathcal{R} \equiv 1 + ht \mod h^2$, which satisfy the pentagon and hexagon identities and have the following symmetries.*

$$\Phi^{321}\Phi = 1 \qquad \mathcal{R}^{21} = \mathcal{R}, \tag{A.1}$$

$$\Phi^{\theta} = \Phi \qquad \mathcal{R}^{\theta} = \mathcal{R}, \tag{A.2}$$

$$\Phi^{S}\Phi = 1 \qquad \mathcal{R}^{S} = \mathcal{R}, \tag{A.3}$$

$$\hat{\Phi} = \Phi \qquad \hat{\mathcal{R}}\mathcal{R} = 1. \tag{A.4}$$

Proof In [Dr2] Drinfeld proved the existence of a pair $(\Phi, \mathcal{R})$ satisfying the pentagon and hexagon identities using standard Cartier coalgebra coboundary, δ, to study the pentagon equation and a modified Cartier coboundary, δ', in which the last factor is "frozen" to study the hexagon identities. In the proof he imposes the condition (A.1). In fact, the remaining conditions can be included in his proof. We shall prove this by showing that the crucial obstruction equations can be solved in the appropriate subcomplex.

As usual, suppose that we have a pair $(\Phi, \mathcal{R})$ giving a truncated deformation defined to order h^n and we want to extend it to order h^{n+1}. (All further equations which contain h^n will be understood to be modulo h^{n+1}.) Define the obstruction pair (ξ, ψ) by

$$\mathrm{Pent}(\Phi) = 1 + \xi h^n \mod h^{n+1} \tag{A.5}$$

$$(\Delta \otimes id)\mathcal{R} = \Phi^{312}\mathcal{R}^{13}(\Phi^{132})^{-1}\mathcal{R}^{23}\Phi + \psi h^n \mod h^{n+1} \tag{A.6}$$

Assuming $\mathcal{R}$ symmetric and $\Phi^{321}\Phi = 1 \bmod h^{n+1}$, transposing tensor factors 1 and 3, (A.6) gives the other hexagon identity with error term (obstruction cochain) ψ^{321}.

By the discussion in §2 we know that there is an extension

$$\Phi' = \Phi + \phi h^n,$$

such that the pentagon identity together with all the identities (A.1-3) are satisfied to order h^{n+1}. The element ϕ can be modified by a δ cocycle with the appropriate symmetries. The condition (A.4), which did not appear in §2, is trivial at this stage, since we can simply assume that the deformation parameter is h^2. This means that the obstruction ξ is automatically zero at odd orders. However, when we turn to the hexagon identities, the obstruction ψ is not necessarily zero at odd order and Drinfeld uses the freedom of adding a δ cocycle to Φ in "killing the obstruction". We must show that when we impose the extra symmetries, no modification of Φ is necessary at odd orders. More explicitly, if we extend Φ as above and $\mathcal{R}$ by

$$\mathcal{R}' = \mathcal{R} + rh^n,$$

then the new obstruction is

$$\psi - (\Delta \otimes id)r + \phi^{312} + r^{13} - \phi^{132} + r^{23} + \phi.$$

The terms in r define the modified Cartier operator $\delta' r = (\Delta \otimes id)r - r^{13} - r^{23}$. Transposing factors 1,2 and subtracting gives

$$\psi - \psi^{213} + 6\,\mathrm{Alt}\,\phi.$$

Drinfeld proves that $\psi - \psi^{213} \in \wedge^3\mathcal{G}$, and it is possible to cancel this term by adding a δ cocycle to Φ (since $\delta(\wedge^3\mathcal{G}) = 0$). We want to show that for n odd, $\psi - \psi^{213} = 0$. At this point the additional conditions (A.3-4) on $\mathcal{R}$ become relevant. Suppose that these conditions are satisfied modulo h^n, and define χ by

$$\hat{\mathcal{R}}\mathcal{R} = 1 + \chi h^n \tag{A.7}$$

Since $\mathcal{R} = 1 \bmod h$ we also have

$$\mathcal{R}\hat{\mathcal{R}} = 1 + \chi h^n. \tag{A.8}$$

Substituting $-h$ for h we find

$$\hat{\mathcal{R}}\mathcal{R} = 1 + \chi(-h)^n.$$

Therefore, $\chi = 0$ for n odd. For n even, applying S to (A.8), we get

$$(\mathcal{R})^S(\hat{\mathcal{R}})^S = 1 + \chi^S h^n,$$

so $\chi^S = \chi$. Setting

$$\mathcal{R}' = \mathcal{R} - \frac{1}{2}\chi h^n,$$

we have (A.3-4) to order h^{n+1}. We compute the obstruction ψ for this $\mathcal{R}'$, which we shall denote simply $\mathcal{R}$. Applying S,

$$\begin{aligned}(\Delta \otimes id)\mathcal{R} = (\Delta \otimes id)(\mathcal{R})^S &= (\Phi^{312}\mathcal{R}^{13}(\Phi^{132})^{-1}\mathcal{R}^{23}\Phi)^S + \psi^S h^n \\ &= \Phi^S(\mathcal{R}^{23})^S((\Phi^{132})^{-1})^S(\mathcal{R}^{13})^S((\Phi^{312})^S + \psi^S h^n \\ &= \Phi^{-1}\mathcal{R}^{23}\Phi^{132}\mathcal{R}^{13}(\Phi^{312})^{-1} + \psi^S h^n \\ &= \Phi^{321}\mathcal{R}^{23}(\Phi^{231})^{-1}\mathcal{R}^{13}\Phi^{213} + \psi^S h^n,\end{aligned}$$

where we used $\Phi^{321} = \Phi^{-1}$ in the last step. Transposing factors 1,2 leaves the left side invariant and gives

$$(\Delta \otimes id)R = \Phi^{312}\mathcal{R}^{13}(\Phi^{132})^{-1}\mathcal{R}^{23}\Phi + (\psi^{213})^S h^n.$$

Therefore

$$(\psi^{213})^S = \psi. \tag{A.9}$$

On the other hand

$$\begin{aligned}((\Delta \otimes id)\hat{\mathcal{R}})^S &= (\hat{\Phi}^{312}\hat{\mathcal{R}}^{13}(\hat{\Phi}^{132})^{-1}\hat{\mathcal{R}}^{23}\hat{\Phi})^S + \psi^S(-h)^n \\ &= \Phi^{-1}(\mathcal{R}^{23})^{-1}\Phi^{132}(\mathcal{R}^{13})^{-1}(\Phi^{312}) + (-1)^n\psi^S h^n\end{aligned}$$

Taking inverses in (A.6) gives

$$((\Delta \otimes id)\hat{\mathcal{R}})^S = (\Delta \otimes id)\mathcal{R}^{-1} = (\Phi^{312}\mathcal{R}^{13}(\Phi^{132})^{-1}\mathcal{R}^{23}\Phi + \psi h^n)^{-1}$$
$$= \Phi^{-1}(\mathcal{R}^{23})^{-1}\Phi^{132}(\mathcal{R}^{13})^{-1}(\Phi^{312}) - \psi h^n.$$

Thus

$$\psi^S = (-1)^{n+1}\psi. \tag{A.10}$$

Then

$$\psi^{213} = \psi^S = (-1)^{n+1}\psi,$$

so ψ is symmetric in 1,2 for n odd, and antisymmetric in 1,2 for n even.

For n even we can "kill the obstruction" by adding ϕh^n to Φ where $\phi = \frac{1}{3}\psi \in \wedge^3\mathcal{G}$. For n odd, we don't change Φ but add to $\mathcal{R}$ a term rh^n satisfying the equation

$$\delta' r = \psi, \tag{A.11}$$

where δ' is the modified Cartier coboundary in dimension 2. The second cohomology group for δ' is $\wedge^2\mathcal{G} \otimes U(\mathcal{G})$, and since ψ is symmetric in 1,2 it must cobound, moreover ψ is S invariant and the cochain it cobounds can be chosen S invariant. Thus the condition (A.3) is preserved. Moreover, since n is odd, and $\mathcal{R} = 1$ mod h, adding any term rh^n will not affect condition (A.4). This shows that one can take the next step in the recursive construction of the pair $(\Phi, \mathcal{R})$. $\square$

Consider the regular representation of $U(\mathcal{G})$ on the the compactly supported functions on the corresponding Lie group G. If we define a scalar product using the Haar measure, then for $u \in U(\mathcal{G})$ the operator $S(u)$ is a formal adjoint to the operator u. The elements F, R of the double tensor product and Φ of the triple tensor product can be considered as formal power series operators acting on the functions on $G \times G$ and $G \times G \times G$ respectively. Extending S conjugate linearly and using the pure imaginary deformation parameter $i\nu$ conditions (A.3) and (A.4) and the conditions (21) and (46) on F imply formal unitarity.

Corollary. *Extending S conjugate linearly and using it to define the formal adjoint, as explained above, we have the following formal unitarity conditions on $F, \mathcal{R}$ and Φ:*

$$\begin{aligned} F_{i\nu}F_{i\nu}^* &= F_{i\nu}F_{-i\nu}^S = 1 \\ (\Phi_{i\nu})^*\Phi_{i\nu} &= \Phi_{-i\nu}^S\Phi_{i\nu} = 1 \\ (\mathcal{R}_{i\nu})^*\mathcal{R}_{i\nu} &= \mathcal{R}_{-i\nu}^S\mathcal{R}_{i\nu} = 1. \end{aligned}$$

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